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Instantons in matrix valued Φ^4 quantum field theory

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Abstract. In scalar ϕ^4 quantum field theory the instanton solution which gives the minimal action is a spherically symmetric solution due to the proof of the Sobolev inequality for scalar functions. The value of the minimal action is related to the Sobolev constant of the Sobolev inequality.

We generalise this to matrix valued Φ^4 field theory by generalising the Sobolev inequality to matrix valued functions.

1. Introduction

Consider a massless ϕ^4 field theory in four dimensions. The correlation functions are given by

$$G^N(x_1, \ldots, x_N) = \int \mathrm{D}\phi(x)\phi(x_1), \ldots, \phi(x_N) \exp(-A(\phi)) \tag{1.1}$$

where the action $A(\phi)$ is

$$A(\phi) = \int \mathrm{d}^4x[\tfrac{1}{2}(\partial_\mu\phi)^2 + \tfrac{1}{4}g\phi^4] \tag{1.2}$$

and ϕ is a scalar spin-zero field, the above formulation being Euclidean.

For instanton solutions, we have to look for non-trivial solutions with finite action to the Euclidean field equation with g negative.

The Euclidean field equation is

$$-\Delta\phi(x) + g\phi^3(x) = 0. \tag{1.3}$$

Set

$$\phi(x) = (1/\sqrt{-g})f(x) \tag{1.4}$$

then $f(x)$ satisfies

$$-\Delta f(x) - f^3(x) = 0 \tag{1.5}$$

(1.5) being a numerical equation.

It can be shown, using the Sobolev inequality (Stein 1970, Klauder 1973) for scalar valued functions, that solutions leading to the minimal action are spherically symmetric. If we choose an origin x_0 and define $r = |x - x_0|$ then we have

$$\left[-\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^2 - \frac{(d-1)}{r}\frac{\mathrm{d}}{\mathrm{d}r} + 1\right]f(x) - f^3(x) = 0 \tag{1.6}$$

where $f(x)$ for this equation gives the minimal action. If we find a solution to equation (1.5), which is a numerical number, A, then the classical action is given by

$$A(\phi_c) = -A/g \tag{1.7}$$

by looking at equations (1.4) and (1.2). If $A(\phi_c)$ is finite then so is $A(\lambda\phi_c)$ and $A[\lambda\phi_c]$ is stationary at $\lambda = 1$. This gives

$$\int d^4x[(\partial_\mu\phi_c)^2 + g\phi_c^4] = 0 \tag{1.8}$$

then

$$A(\phi_c) = \int d^4x(-\tfrac{1}{2}g\phi_c^4 + \tfrac{1}{4}g\phi_c^4) \tag{1.9}$$

$$= -\frac{1}{4g}\int d^4x f^4(x) \tag{1.10}$$

$$A = \tfrac{1}{4}\int f^4(x)\, d^4x > 0. \tag{1.11}$$

Following Parisi (1977) consider

$$R(\phi) = \frac{(\int d^4x(\partial_\mu\phi)^2)^2}{\int \phi^4\, d^4x}. \tag{1.12}$$

If dimension d is smaller or equal to 4, $R(\phi)$ is bounded from below by constant R

$$R(\phi) \geqslant R \geqslant 0 \tag{1.12$'$}$$

and there exists a spherically symmetric zero free function $\phi_c(x)$ which saturates the bound $R(\phi_c) = R$ and is a solution of the variational equation

$$\delta R(\phi)/\delta\phi(x) = 0 \tag{1.13}$$

then this can be explicitly written as

$$-\Delta\phi_c(x) - \phi_c^3(x)K = 0 \tag{1.14}$$

where

$$K = \frac{\int d^4x(\partial_\mu\phi_c^2)}{\int \phi_c^4\, d^4x}. \tag{1.15}$$

If we rescale $\phi_c(x)$, then with $K = 1$ we have

$$-\Delta\phi_c(x) - \phi_c^3(x) = 0 \tag{1.16}$$

equation (1.16) being identical to (1.5).

For each instanton solution, we had derived the identity (1.8) which for $f(x)$ can be written as

$$\int d^4x(\partial_\mu f(x))^2 = \int d^4x f^4(x). \tag{1.17}$$

As

$$A = \tfrac{1}{4}\int d^4x f^4(x) = \frac{1}{4}\frac{(\int d^4x f^4(x))^2}{\int d^4x f^4 x} \tag{1.18}$$

we can write the action $A(\phi_c)$ using (1.17) and (1.18) as

$$A(\phi_c) = -\frac{1}{4g}\frac{(\int(\partial_\mu f(x))^2\,\mathrm{d}^4x)^2}{\int f^4(x)\,\mathrm{d}^dx}. \tag{1.19}$$

The smallest action therefore corresponds to the minimum of $R(\phi)$. Thus

$$A = \tfrac{1}{4}R \tag{1.20}$$

and the solution $f(x)$ we are looking for is given by $f(x) = \phi_c(x)$ for $K = 1$.

The problem of finding the minimal action is now reduced to finding the minimum of $R(\phi)$ and the corresponding solution $\phi_c(x)$. We discuss this in a more mathematical approach, which will help us later when we generalise it to matrix valued functions.

2. Sobolev inequality of scalar valued functions

Let $f(x)$ be scalar valued function on $\mathbb{R}^e$. We define the Sobolev norm on $f(x)$ as

$$\|f(x)\|_p = \left(\int_{\mathbb{R}^e}\mathrm{d}^ex|f|^p\right)^{1/p} \tag{2.1}$$

where $|f| = (f^+f)^{1/2}$ (note f^+ is the complex conjugate of f). We write the general inequality as

$$\|\Delta f\|_p \geqslant C_{e,p}\|f\|_{\bar{p}} \tag{2.2}$$

where

$$\frac{1}{\bar{p}} = \frac{1}{p} - \frac{1}{e} \tag{2.3}$$

$C_{e,p}$ is the sharp (or best) constant of the above inequality.

Theorem. If f and $\Delta f \in \mathbb{R}^e$ and $e > 2$ then

$$\|\Delta f\|_2 \geqslant C_{e,2}\|f\|_{\bar{2}} \qquad \bar{2} = \frac{2e}{e-2} \tag{2.4}$$

where

$$C_{e,2} = \left(\Pi e(e-2)^{-e/2}\frac{\Gamma(e)}{\Gamma(e/2)}\right)^{-1/4} \tag{2.5}$$

and the maximising function $f(x)$ which gives the above Sobolev constant is

$$f(x) = (a + br^2)^{2-e/2}. \tag{2.6}$$

We note that the Sobolev inequality is a special type of the general inequality, the value of p being 2.

We also note that when $e = 4$ (in field theory e will be considered to be the dimension) the Sobolev inequality (2.4) becomes related to equation (1.12) such that

$$(C_{4,2})^4 = R = 4A$$

using (2.4), (1.12), (1.12′) and (1.20).

The proof of the above theorem can be found in Stein (1970), Klauder (1973) and Lieb (1983). However we write it, it involves the idea of Schwarz symmetrisation (Lieb 1983) and classical rearrangement inequalities (Hardy *et al* 1952).

3. Sobolev inequality of matrix valued functions

As shown in the previous section the Sobolev inequality gives the minimal action to an instanton in ϕ^4 field theory. Now many quantum field theories possess natural generalisation in which the number of degrees of freedom is a free parameter, the ϕ^4 field theory being one of them. We shall therefore generalise the Sobolev inequality to matrix valued functions $F(x)$, which will help us in giving the minimal action to an instanton solution in Φ^4 field theory, Φ being matrix valued.

To achieve this, first the idea of Scharz symmetrisation and the classical rearrangement inequality has to be extended to matrix valued functions.

4. Holder inequality and Schwarz symmetrisation

On the linear space $M_s(\mathbb{R}^e)$ of $s \times s$ measurable complex matrix valued functions on $\mathbb{R}^e$, we define the following norms:

$$\|F\|_p = \left(\int_{\mathbb{R}^e} d^e x \, \mathrm{Tr}(|F(x)|^p) \right)^{1/p} \tag{4.1}$$

$$\|F\|_\infty = \sup_{x \in \mathbb{R}^e} \|F(x)\| \tag{4.2}$$

where $|A|$ is $(A^+ A)^{1/2}$ using the polar decomposition $A = U|A|$ of A into a unitary matrix U and a positive semidefinite factor $|A|$. In particular, we have

$$\|F\|_2 = \left(\int_{\mathbb{R}^e} d^e x \, \mathrm{Tr}(F^+(x) F(x)) \right)^{1/2}. \tag{4.3}$$

Note that a dagger is used to denote the adjoint matrix. In this paper the asterisk will always denote Schwarz symmetrisation, defined later. Clearly the subspace $M_{s,p}(\mathbb{R}^e)$ of $M_s(\mathbb{R}^e)$ for which the norm $\|\cdot\|_p$ is finite forms a Banach space under this norm.

By combining the standard Holder inequality for complex functions with the corresponding matrix inequality

$$\mathrm{Tr}(|AB|^p)^{1/p} \leqslant (\mathrm{Tr}(|A|^q))^{1/q} (\mathrm{Tr}(|B|^r))^{1/r} \qquad \frac{1}{p} = \frac{1}{q} + \frac{1}{r} \tag{4.4}$$

we get the Holder inequality for $M_s(\mathbb{R}^e)$:

$$\|FG\|_p \leqslant \|F\|_q \|G\|_r \qquad \frac{1}{p} = \frac{1}{q} + \frac{1}{r}. \tag{4.5}$$

We can also extend the notion of Schwarz symmetrisation to $M_s(\mathbb{R}^e)$.

Definition 4.1. Let F be a function in $M_s(\mathbb{R}^e)$ and let the $\sigma_r(x)$, $1 < r < s$, be the associated singular value functions chosen in decreasing order

$$\sigma_1(x) \geqslant \sigma_2(x) \geqslant \ldots \geqslant \sigma_s(x) \geqslant 0. \tag{4.6}$$

Then a function F^* in $M_s(\mathbb{R}^e)$ is a Schwarz symmetrisation of F if it satisfies the following.

(a) $F^*(x)$ is a diagonal matrix $\mathrm{diag}(\sigma_1^*(x), \ldots, \sigma_s^*(x))$.

(b) $F^*(x_1) = F^*(x_2)$ when $|x_1| = |x_2|$.

(c) $0 \leqslant |x_1| \leqslant |x_2| \Rightarrow F^*(x_1) \geqslant F^*(x_2)$.

(d) $\mu(\{x: \sigma_r^*(x) > c\}) = \mu(\{x: \sigma_r(x) > c\})\ \forall c > 0$ and $1 \leqslant r \leqslant s$.

Remark. Immediate consequences of this definition are

(i) for $s=1$ it reduces to the conventional definition (Lieb 1983),

(ii) $\sigma_1^*(x) \geqslant \sigma_2^*(x) \geqslant \ldots \geqslant \sigma_s^*(x) \geqslant 0$,

(iii) if $U(x)$ is unitary matrix valued then $(UFU^\top)^* = F^*$,

(iv) if $F \in M_2(\mathbb{R}^e)$ then so is F^* and $\|F\|_p = \|F^*\|_p$ for $1 \leqslant p \leqslant \infty$.

5. Rearrangement inequalities

The basic integral rearrangement inequality, originally due to Hardy *et al* (1952), has been strengthened by Brascamp *et al* (1974) to the following form.

Theorem 5.1. Let f_j, $1 \leqslant j \leqslant k$ be complex measurable functions on $\mathbb{R}^e$ and let a_{jm}, $1 \leqslant j \leqslant k$, $1 \leqslant m \leqslant n$ be real numbers. Then

$$\int_{\mathbb{R}^{ne}} \mathrm{d}^{ne}x \left| \prod_{j=1}^{k} f_j\left(\sum_{m=1}^{n} a_{jm} x_m \right) \right| \leqslant \int_{\mathbb{R}^{ne}} \mathrm{d}^{ne}x \prod_{j=1}^{k} f_j^*\left(\sum_{m=1}^{n} a_{jm} x_m \right). \tag{5.1}$$

In this section we prove the following extension.

Theorem 5.2. Let F_j, $1 \leqslant j \leqslant k$ be measurable $s \times s$ matrix valued functions on $\mathbb{R}^e$ and let a_{jm}, $1 \leqslant j \leqslant k$, $1 \leqslant m \leqslant n$ be real numbers. Then

$$\int_{\mathbb{R}^{ne}} \mathrm{d}^{ne}x \,\mathrm{Tr}\left[\left| \prod_{j=1}^{k} F_j\left(\sum_{m=1}^{n} a_{jm} x_m \right) \right| \right] \leqslant \int_{\mathbb{R}^{ne}} \mathrm{d}^{ne}x \,\mathrm{Tr}\left[\prod_{j=1}^{k} F_j^*\left(\sum_{m=1}^{n} a_{jm} x_m \right) \right]. \tag{5.2}$$

Remark. The inequality holds for all possible orderings of the matrix product under the trace operation and so no particular order is specified.

We need the following lemma of Fan (Marshall and Olkin 1979) in the proof of theorem 5.2.

Lemma 5.3. Let T_j, $1 \leqslant j \leqslant k$ be a set of complex $s \times s$ matrices with singular values $\sigma_{1j} \geqslant \sigma_{2j} \geqslant \ldots \sigma_{sj} \geqslant 0$ and let $D_j = \mathrm{diag}\{\sigma_{ij}, \ldots, , \sigma_{sj}\}$, $1 \leqslant j \leqslant k$ be the corresponding singular value diagonalisations. If U_j, $1 \leqslant j \leqslant k$ are unitary $s \times s$ matrices then

$$|\mathrm{Tr}(U_1 T_1 U_2 T_2 \ldots U_k T_k)| \leqslant \mathrm{Tr}(D_1 D_2 \ldots D_k).$$

Corollary 5.4.

$$\mathrm{Tr}(|T_1 T_2 \ldots T_k|) \leqslant \mathrm{Tr}(D_1 D_2 \ldots D_k).$$

Proof of theorem 5.2. On the left-hand side of (5.2) we can replace the F by their decreasing singular value diagonalisations as defined in lemma 5.3. Corollary 5.4 guarantees that the value of the integral is not decreased. The proof is then completed by applying theorem 5.1 to each term in the sum defining the trace.

6. Sobolev inequality

A feature of Schwarz symmetrisation which is useful in proving Sobolev inequalities is the global smoothing property. This is illustrated by the next lemma, an extension of one due to Lieb (1983).

Lemma 6.1. Let F and $\nabla F \in M_{s,2}(\mathbb{R}^e)$. Then $\nabla F^* \in M_{s,2}(\mathbb{R}^e)$ and

$$\|\nabla F\|_2 \geq \|\nabla F^*\|_2. \tag{6.1}$$

Remark. The term $\|\nabla F\|_2$ is a concise notation for

$$\left(\sum_{i=1}^{e} \left\|\frac{\delta F}{\delta x_i}\right\|_2^2\right)^{1/2}.$$

Proof. Let Δ be the Laplacian in e dimensions. Then $e^{t\Delta}$, for $t>0$, has a Gaussian kernel $h_t(x-y)$, which is Schwarz symmetric. We can also define the Schwarz symmetric matrix valued kernel

$$H_t(x-y) = h_t(x-y)I.$$

Using theorem 5.2, we have

$$\int_{\mathbb{R}^{2e}} d^e x\, d^e y\, \mathrm{Tr}(G^+(x)H_t(x-y)G(y)) \leq \int_{\mathbb{R}^{2e}} d^e x\, d^e y\, \mathrm{Tr}(G^*(x)H_t(x-y)G^*(y)) \tag{6.2}$$

for any F, $G \in M_{s,2}(\mathbb{R}^e)$ and $t>0$. Since $\{e^{t\Delta}, t>0\}$ is a strongly continuous contraction semigroup, we have

$$\lim_{t\to 0} \frac{1}{t} \int_{\mathbb{R}^{2e}} d^e x\, d^e y\, \mathrm{Tr}(F^+(x)(H_t(x-y)F(y) - F(x))) = \int_{\mathbb{R}^e} d^e x\, \mathrm{Tr}(F^+(x)\,\Delta F(x)) \tag{6.3}$$

for sufficiently smooth F of compact support. Applying to each side of (6.2) and integrating by parts, we obtain

$$-\int_{\mathbb{R}^e} d^e x \sum_{i=1}^{e} \mathrm{Tr}\left(\frac{\delta G^+(x)}{\delta x_i}\frac{\delta G(x)}{\delta x_i}\right) \leq -\int_{\mathbb{R}^e} d^e x \sum_{i=1}^{e} \mathrm{Tr}\left(\frac{\delta G^*(x)}{\delta x_i}\frac{\delta G^*(x)}{\delta x_i}\right)$$

from which the theorem follows by extension to the domain of ∇.

We do not know of a correspondingly simple proof of the more general inequality $\|\nabla F\|_p \geq \|\nabla F^*\|_p$. The Sobolev constant $C_{e,p,r}$ is defined as the infimum of the ratio $\|\nabla F\|_p/\|F\|_{\bar{p}}$ where $1/\bar{p} = 1/p - 1/e$ for $1 \leq p \leq e$.

In the case of $p=2$, we have the following theorem.

Theorem 6.2. If F and $\nabla F \in M_{s,2}(\mathbb{R}^e)$ and $e>2$, then

$$\|\nabla F\|_2 \geq C_{e,2s}\|F\|_{\bar{2}} \qquad \bar{2} = 2e/e-2 \tag{6.4}$$

where

$$C_{e,2,s} = C_{e,2,1} = [(\Pi e(e-2))^{-e/2}\Gamma(e)/\Gamma(e/2)]^{-1/4}. \tag{6.5}$$

Proof. From lemma 6.1, an optimising function must be Schwarz symmetric and so of the form $\mathrm{diag}\{\sigma_1(x), \ldots, \sigma_s(x)\}$. The Euler equation for stationary values of the functional

$$J(F) = \|\nabla F\|_2/\|F\|_{\bar{2}}$$

is, for this class of functions,

$$\frac{\Delta|F|}{\|\nabla F\|_2^2} + \frac{|F|^{\bar{2}-1}}{\|F\|_{\bar{2}}^{\bar{2}}} = 0. \tag{6.6}$$

We can rescale F so that $\|\nabla F\|_2^2 = \|F\|_{\bar{2}}^{\bar{2}}$, giving

$$\Delta|F| + |F|^{\bar{2}-1} = 0. \tag{6.7}$$

The matrix equation (6.7) decomposes into the decoupled equations

$$\Delta\sigma_i(x) + (\sigma_i(x))^{\bar{2}-1} = 0 \qquad 1 \leqslant i \leqslant S \tag{6.8}$$

in which the σ_i are non-increasing functions of $|x|$. The general solution to this equation is well known (Talenti 1976) to be

$$\sigma_i(x) = (a_i + b_i|x|^2)^{1-e/2} \qquad 1 \leqslant i \leqslant s \tag{6.9}$$

where a_i, $b_i > 0$.

A scaling argument can be used to show that, for

$$S_{a,b}(x) = (a + b|x|^2)^{1-e/2}$$

$$\int \mathrm{d}^e x (S_{a,b}(x))^{\bar{2}} = (ab)^{-e} \int \mathrm{d}^e x (S_{1,1}(x))^{\bar{2}}$$

$$\int \mathrm{d}^e x (\nabla S_{a,b}(x))^2 = (ab)^{-(e-2)/2} \int \mathrm{d}^e x (\nabla S_{1,1}(x))^2.$$

Hence, for functions of the form (6.9), we have

$$\frac{\|\nabla F\|_2}{\|F\|_{\bar{2}}} = \left(\sum_{i=1}^{s} (a_i b_i)^{-(e-2)/2}\right)^{1/2} \left[\left(\sum_{i=1}^{s} (a_i b_i)^{-e}\right) \frac{e-2}{2e}\right]^{-1} C_{e,2}$$

where $C_{e,2}$ is the known best constant for the case $s = 1$. The factor in front of $C_{e,2}$ is $\geqslant 1$, by Jensen's inequality. The infimum is attained for functions F, in which only one matrix element, on the diagonal, is non-zero and this element is of the form (5.9). More generally, we can take any fixed one-dimensional orthogonal projector multiplied by the same scalar function of x.

We can generalise the Sobolev inequality to the case of rectangular matrices. On the linear space $M_{st}(\mathbb{R}^e)$ of $s \times t$ measurable complex matrix valued functions on $\mathbb{R}^e$, we define the norm as before, i.e.

$$\|F\|_p = \left(\int_{\mathbb{R}^e} \mathrm{d}^e x \, \mathrm{Tr}(|F(x)|^p)\right)^{1/p}$$

where

$$|F(x)| = (F^+(x)F(x))^{1/2}.$$

Without loss of generality, we can take $F(x)$ to have more columns than rows (if otherwise, $F(x)$ can be interchanged with $F(x)$, leaving the norm invariant).

Theorem 6.3. If F and $\nabla F \in M_{st,2}(\mathbb{R}^e)$, then for $e > 2$

$$\|\nabla F\|_2 \geqslant C_{e,2,st}\|F\|_{\bar{2}} \qquad \bar{2} = \frac{2e}{e-2} \tag{6.10}$$

where

$$C_{e,2,st} = C_{e,2,1} = \left([\pi e(e-2)]^{-e/2} \frac{\Gamma(e)}{\Gamma(e/2)}\right)^{-1/4}. \tag{6.11}$$

Proof. Write

$$C_{e,2,st} = \inf_F \frac{\|\nabla F\|_2}{\|F\|_{\bar{2}}}. \tag{6.12}$$

Apply the following transformation on F, giving F':

$$F' = N_{ts}F_{st} \qquad F \equiv F_{st} \qquad N \equiv N_{ts} \tag{6.13}$$

where N_{ts} is a $t \times s$ rectangular matrix with the property $N^+N = I$ and N_{ts} is independent of x. Then $N_{ts}F_{st}$ is a square matrix. The norm of F and the norm of the derivative of F stay invariant under the above transformation, i.e.

$$\|F'\| = \|F\| \qquad \text{and} \qquad \|\nabla F'\| = \|\nabla F\|.$$

Restricting F to the subclass of positive semidefinite matrices, the Euler equation for stationary values of the functional

$$J(F) = \|\nabla F\|_2 / \|F\|_{\bar{2}} \tag{6.14}$$

is $\Delta|F| + |F|^{\bar{2}-1} = 0$, as before, the above equation being identical to equation (4.7). Hence

$$C_{e,2,st} = C_{e,2,1}. \tag{6.15}$$

7. Conclusion

When considering a massless Φ^4 field theory, in four dimensions Φ being a matrix valued (any rectangular matrix), when the action $A(\Phi)$ is

$$A(\Phi) = \int d^4x[\tfrac{1}{2}(\partial_\mu \Phi)^2 + \tfrac{1}{4}g\Phi^4]$$

then (6.15) tells us that the minimal action of the instanton solution in the matrix valued Φ^4 field theory is the same as in the scalar valued ϕ^4 field theory. Further it can also be shown that in the massive case no instanton solution exists in four dimensions for the minimal action but the minimal action is the same as that for the massless case.

References

Brascamp H J, Lieb E H and Luttinger J M 1974 *J. Funct. Anal.* **17** 227

Hardy G E, Littlewood J E and Polya G 1952 *Inequalities* (Cambridge: Cambridge University Press)

Klauder J R 1973 *Acta Phys. Aust. Suppl.* **11** 341

Lieb E H 1983 *Ann. Math.* **118** 349

Marshall A W and Olkin I 1979 *Inequalities: Theory Of Majorisation And Its Applications* (New York: Academic) p 516

Parisi G 1977 *Phys. Lett.* **66B** 167

—— 1977b *Phys. Lett.* **68B** 361

Stein E M 1970 *Singular Integrals And Differentiability Properties Of Functions* (Princeton, NJ: Princeton University Press)

Talenti G 1976 *Ann. Matem. Pura. App.* **110** 353